International Multidisciplinary Innovative Research Journal
-An International refereed e-journal

ISSN: 2456 – 4613
Volume – IV Issue - 1
November - 2019

A STUDY ON PRIMITIVE IDEMPOTENTS IN SEMICENTRAL SEMINEAR-RINGS

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ABSTRACT:

In this paper, we made an extended study on semicentral seminear-rings. It is proved that left (right) semicentral seminear-ring, $S_l(R)$ $(S_r(R))$ is commuting if and only if it is central seminear-ring. Also, it is showed that a primitive left (right) semicentral seminear-ring is additive if and only if it is orthogonal. It is observed that in a left (right) semicentral seminear-ring, $S_l(R)$ $(S_r(R))$ which is commuting then the left (right) semicentral quotient seminear-ring, $S_l(R/N)$ $(S_r(R/N))$ is orthogonal iff left (right) semicentral seminear-ring, $S_l(R)$ $(S_r(R))$ is orthogonal. The same concept for primitive semicentral seminear-ring is also proved.

Mathematics Subject Classification: 16Y30

Keywords:

Additive, Commuting, Left (right) primitive semicentral idempotent and Left (right) semicentral seminear-ring.

INTRODUCTION:

In a semiring (N, +, .) if we ignore commutativity of + and one distributive law, (N, +, .) is a seminear-ring. If we do not stipulate the left distributive law, (N, +, .) is a right seminear-ring. Wily G Van Hoorn and Van Rootselaar [21] introduced the notion of seminear-rings. Seminear-rings are the generalization of semi-ring and near-rings. Especially he discussed homomorphisms in seminear-rings and obtained some interesting properties. Further, many people [1, 9, 10, 5, 7, 8, 20] worked in the field of seminear-rings and explored many interesting results. In 2018, we [16] introduced the notion of semicentral seminear-rings. In this paper, we discussed some more properties of orthogonal elements in semicentral seminear-rings.

PRELIMINARIES:

DEFINITON: 2.1 [10]

A *right seminear-ring* is a non-empty set R together with the binary operations '+' and '.' such that (R, +) and (R,.) are semigroup and $(a + b).c = (a.c) + (b.c)$ holds for all a, b, c in R.

In the same way, we may define a left seminear-ring in which left distributive law holds instead of right distributive law.

DEFINITION: 2.2 [7]

A right seminear-ring R is said to have an *absorbing zero* if

(i) $a + 0 = 0 + a = a$

(ii) $a.0 = 0.a = 0$ holds for all a in R

DEFINITION: 2.3 [7] An element a in the seminear-ring R is called an *idempotent* if $a^2 = a$

DEFINITION: 2.4 [7]

An idempotent e in the seminear-ring R is said to be *central* if $ex = xe$ holds for all x in the seminear-ring R.

DEFINITION: 2.5 [7]

A seminear-ring R is said to be *central seminear-ring* if every idempotent in R is central.

DEFINITION: 2.6 [16]

An element a in the seminear-ring R is said to be *nilpotent* if there exists a positive integer k such that $a^k = 0$.

DEFINITION: 2.7 [16]

An idempotent e in the seminear-ring R is called *left semicentral* if $Re = eRe$

DEFINITION: 2.8 [16]

An idempotent e in the seminear-ring R is called *right semicentral* if $eR = eRe$

DEFINITION: 2.9 [16]

A seminear-ring R in which every idempotent is left (right) semicentral is called *left (right) semicentral seminear-ring.*

NOTATION: 2.10 [16]

(i) E denotes the set of all idempotents in the seminear-ring R

(ii) C(R) is the set of all central idempotents in the seminear-ring R

(iii) $S_l(R)$ denotes the set of all left semicentral elements in the seminear-ring R

(iv) $S_r(R)$ denotes the set of all right semicentral elements in the seminear-ring R

PROPOSITION: 2.11 [16]

Any idempotent e in the seminear-ring R is left semicentral if and only if $1 - e$ in R is right semicentral.

PROPOSITION: 2.12 [16]

For any idempotent e in the seminear-ring R, TFAE:

(i) e is left semicentral

(ii) xe is an idempotent for all idempotents $x \in$ R

(iii) $xe = exe$, for all idempotents $x \in$ R

(iv) $(xe)^n = (exe)^n$, for all idempotents $x \in$ R

COROLLARY: 2.13 [16]

For any idempotent e in the seminear-ring R, TFAE:

(i) e is right semicentral

(ii) ex is an idempotent for all idempotents $x \in$ R

(iii) $ex = exe$, for all idempotents $x \in$ R

(iv) $(ex)^n = (exe)^n$, for all idempotents $x \in$ R

SOME RESULTS ON PRIMITIVE IDEMPOTENTS IN SEMICENTRAL SEMINEAR-RINGS

DEFINITION: 3.1

A subset S of a seminear-ring R is called *commuting* if $ef = fe$ for all e, f in S

DEFINITION: 3.2

Two idempotents e, f of the seminear-ring R are said to be *orthogonal* if $ef = fe = 0$

DEFINITION: 3.3

An idempotent e of a seminear-ring R is said to be *primitive* if it cannot be written as a sum of two non-zero orthogonal idempotents.

NOTATION: 3.4

(i) M(R) denotes the set of all primitive idempotents in the seminear-ring R

(ii) $M_l(R) = M(R) \cap S_l(R)$ (i.e.) it denotes the set of all left primitive semicentral elements in the seminear-ring R.

(iii) $M_r(R) = M(R) \cap S_r(R)$ (i.e.) it denotes the set of all right primitive semicentral elements in R.

(iv) J(R) denotes the set of all nilpotent elements in R such that it has no non-zero nilpotent element.

DEFINITION: 3.5

A subset S of E is said to be *additive* in E if for all e, f in S ($e \neq f$), $e + f \in E$.

PROPOSITION: 3.6

Let R be a seminear-ring. Then $S_l(R)$ is additive in E if and only if $S_l(R)$ is commuting and $2ef = 0$ for all e, f in S $(e \neq f)$

Proof:

Suppose that $S_l(R)$ is additive in E. Let $e, f \in S_l(R)$ and $e \neq f$ be arbitrary. Since $S_l(R)$ is additive in E,$e + f \in E$. Thus $e + f = (e + f)^2 = (e + f)(e + f) = e^2 + ef + fe + f^2 = e + f + ef + fe$. This gives $ef = -fe$. Also, $ef = eef = e(ef) = e(-fe) = -(ef)e = (fe)e = f(ee) = fe$. Hence $S_l(R)$ is commuting and also $2ef = 0$. Conversely, suppose $S_l(R)$ is commuting and $2ef = 0$ for all e, f in S and $e \neq f$. Now, $(e + f)^2 = (e + f)(e + f) = e^2 + ef + fe + f^2 = e + ef + fe + f = e + 2ef + f = e + f$. Thus $e + f \in E$ for all e, f in S and $e \neq f$. Hence $S_l(R)$ is additive in E.

COROLLARY: 3.7

Let R be a seminear-ring. Then $S_r(R)$ is additive in E if and only if $S_r(R)$ is commuting and $2ef = 0$ for all e, f in S $(e \neq f)$

Proof:

The proof is similar to the proof of the above proposition: 3.6

LEMMA: 3.8

For a seminear-ring R, IFAE:

(i) $S_l(R)$ is commuting

(ii) $S_r(R)$ is commuting

(iii) $S_l(R)$ = C(R)

(iv) $S_r(R)$ = C(R)

Proof:

(1)⟹(2) follows from proposition: 2.11.(3)⟹(1) and (4)⟹(1) are obvious. Now to prove (1)⟹(3). Assume that $S_l(R)$ is commuting. Let $e \in S_l(R)$ and $a \in R$. Let $f = e + (1 - e)ae$. Then $f \in S_l(R)$. Now, $fe = (e + (1 - e)ae)e = e^2 + (1 - e)ae^2 = e + (1 - e)ae = f$ and $ef = e(e + (1 - e)ae) = e^2 + e(1 - e)ae = e + (e - e^2)ae = e$. Since $S_l(R)$ is commuting, $e = ef = fe = f = e + (1 - e)ae$ and so $ae = eae = ea$. Hence $e \in C(R)$. Thus (1)⟹(3) is proved. In the similar way, (2)⟹(4) follows.

PROPOSITION: 3.9

Let R be a seminear-ring. Then $M_l(R)$ is additive in E if and only if $M_l(R)$ is orthogonal.

Proof:

Suppose $M_l(R)$ is additive in E. Let e, f be in $M_l(R)$such that $ef \neq 0$and $e \neq f$. Since $M_l(R) \subseteq S_l(R)$ and $M_l(R)$ is additive in E, it is commuting. Hence $ef = fe$. Now take $e = ef + (e - ef)$. Then $ef(e - ef) = efe - ef(ef) = efe = effe = efe - efe = 0$ and $(e - ef)ef = eef - ef(ef) = ef - effe = ef -$

$e(fe) = ef - eef = ef - ef = 0$. Since e is primitive and $ef \neq 0, e = ef$. Also, let $f = fe + (f - fe)$. Now, $fe(f - fe) = f(ef) - f(ef)e = ffe - ffee = fe - fe = 0$ and $(f - fe)fe = ffe - f(ef)e = fe - ffee = fe - fe = 0$. Since f is primitive and $ef = fe \neq 0, f = fe$. Thus, $e = ef = fe = f$ which is a contradiction to $e \neq f$. Therefore, $ef = fe = 0$. Hence $M_l(R)$ is orthogonal.

COROLLARY: 3.10

Let R be a seminear-ring. Then $M_r(R)$ is additive in E if and only if $M_r(R)$ is orthogonal.

Proof:

The proof is similar to the proof of the above proposition: 3.9

PROPOSITION: 3.11

Let N ⊆ J(R) be an ideal of a seminear-ring R. If $e, f \in R$ are commuting idempotents such that $\bar{e} = \bar{f} \in R/N$ then $e = f$.

Proof:

Since $\bar{e} = \bar{f} \in R/N, e - f \in N$. And, since $ef = fe$, we have $(e-f)^2 = (e-f)(e-f) = e^2 - ef - fe + f^2 = e - ef - ef + f = e - 2ef + f$. Also, $(e-f)^4 = (e-f)^2(e-f)^2 = (e - 2ef + f)(e - 2ef + f) = e^2 - 2e^2f + ef - 2efe + 4e^2f^2 - 2ef^2 + fe - 2ef^2 + f^2 = e - 2ef + ef - 2ef + 4ef - 2ef + fe - 2ef + f = e - 2ef + f = (e-f)^2$. Therefore, $(e-f)^2$ is an idempotent in R. Thus $(e-f)^2 \in E \cap N \subset E \cap J(R) = \{0\}$ which gives, $(e-f)^2 = e - 2ef + f = 0$. Hence $e + f = 2ef$. By multiplying it with e this yields $e = ef$ and by multiplying the same with f gives $f = ef$. Hence $e - f = ef - ef = 0$. Thus $e = f$.

PROPOSITION: 3.12

Let N ⊆ J(R) be an ideal of R such that idempotents in R/N can be lifted to R. Then we have the following:

(i) $S_l(R)$ is commuting then $S_l(R/N)$ is orthogonal if and only if $S_l(R)$ is orthogonal

(ii) $M_l(R)$ is commuting then $M_l(R/N)$ is orthogonal if and only if $M_l(R)$ is orthogonal.

Proof:

(i) Suppose that $S_l(R/N)$ is orthogonal. Let $e, f \in S_l(R)$ and $e \neq f$. Then clearly $\bar{e}, \bar{f} \in S_l(R/N)$. Assume that $e, f \neq 0$. If $\bar{e} = \bar{f}$, then by previous proposition,$e = f$, which is a contradiction to $e \neq f$. Since $S_l(R/N)$ is orthogonal, $\bar{e}\bar{f} = \bar{f}\bar{e} = \bar{0}$. This implies $ef, fe \in N$, which in turn, is also in E. Thus $ef, fe \in E \cap N \subset E \cap J(R) = \{0\}$. Hence $S_l(R)$ is orthogonal. Clearly, the converse follows.

(ii) If e in R is primitive idempotent, then $\bar{e} \in R/N$ is also primitive. Thus the proof follows from (i).

COROLLARY: 3.13

Let N ⊆ J(R) be an ideal of R such that idempotents in R/N can be lifted to R. Then we have the following:

(i) $S_r(R)$ *is commuting then* $S_r(R/N)$ *is orthogonal if and only if* $S_r(R)$ *is orthogonal*

(ii) $M_r(R)$ *is commuting then* $M_r(R/N)$ *is orthogonal if and only if* $M_r(R)$ *is orthogonal.*

Proof:

The proof is similar to the proof of above proposition: 3.12

PROPOSITION: 3.14

For an idempotent e of a seminear-ring R, TFAE:

(i) *Every* $e \in M_l(R)$ *is central.*

(ii) $M_l(R)$ *is commuting*

(iii) $ef = fe$ *for all* f *in* $M_l(R)$*, which are isomorphic to e*

(iv) $(ef)^n = (fe)^n$ *for all* f *in* $M_l(R)$ *are isomorphic to e where* n *is some positive integer.*

Proof:

(i)⟹(ii) follows from the definition of central. (ii) ⟹(iii) ⟹(iv) follows from the proposition:2.12. So it is enough to prove (iv) ⟹(i). Suppose that the condition (iv) holds and assume that there exists $e \in M_l(R)$ such that $e \notin C(R)$. Then $ea \neq ae$ for some a in R. Consider, $f = e + ea(1-e)$. Then $f \in S_l(R)$ and $e \neq f$. Now, $ef = e\big(e + ea(1-e)\big) = ee + eea(1-e) = e + ea(1-e) = f$ and $fe = \big(e + ea(1-e)\big)e = ee + (ea - eae)e = ee + eae - eaee = e + eae - eae = e$. Thus e is isomorphic to f. Since $eR = eeR = efeR \subseteq efR \subseteq eR$. This gives $eR = efR = fR$ so f is primitive idempotent in R. Therefore, $e = (fe)^n \neq (ef)^n = f$, for any positive integer n, which is a contradiction to (iv). Hence $e \in C(R)$.

COROLLARY: 3.15

For an idempotent e of a seminear-ring R, TFAE:

(i) *Every* $e \in M_{\boldsymbol{r}}(R)$ *is central.*

(ii) $M_r(R)$ *is commuting*

(iii) $ef = fe$ *for all* f*in* $M_r(R)$*, which are isomorphic to e*

(iv) $(ef)^n = (fe)^n$ *for all* f *in* $M_r(R)$ *are isomorphic to e where* n *is some positive integer.*

Proof:

The proof is similar to the proof of above proposition: 2.13& 3.14

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