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APPLICATION OF PARTIAL BELONG RELATION IN W-SOFT SEPARATION AXIOMS AND DECISION MAKING PROBLEMS

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ABSTRACT

In this paper, we investigate w-soft separation axioms and their properties using partial belong relation. Furthermore, the application of partial belong relation to finance-based decision making problems. The results shows that incorporating partial belong relations into w-soft separation axioms not only strengthens foundations of w-soft topology but also provided a flexible tool for handling uncertainty in real-world financial decision making.

Keywords: Soft set, Soft topology, w-soft separation axioms, partial belong relation.

1. INTRODUCTION

In our modern world, the data we encounter is rarely "perfect." Whether we are looking at medical symptoms, financial trends, or engineering specifications, information is often vague, incomplete, or uncertain. Traditional mathematics, which relies on "Yes or No" logic, often fails to capture the Gradients of this uncertainty. To bridge this gap, the concept of Soft Set Theory was introduced. Unlike traditional sets, a soft set allows us to describe objects based on a set of parameters (e.g., instead of just "a house," we describe "a house that is expensive, modern, and near the city"). However, even within soft sets, the way we define whether an element "belongs" to a set can be too rigid. The project addresses the limitations of rigid membership in soft topology by introducing the partial belong relation ($\Subset$), which solves the "natural membership" problem where elements are often forced into binary "in or out" categories. This theoretical shift allows for the development of w-soft separation axioms (T_0 through T_4) that are more flexible, hereditary, and mathematically consistent, providing a refined "grammar" for describing uncertain spaces. These abstract improvements are directly applied to decision-making problems through a specialized

algorithm that filters and ranks alternatives—such as medical treatments or economic investments—by identifying "favorite" soft sets that meet specific parameter thresholds. By achieving these objectives, the project effectively "hits two birds with one stone," bridging the gap between complex abstract topology and practical, high-precision data analysis. Ultimately, this integration proves that partial belonging is a superior tool for modeling the "grey areas" of reality, offering a more inclusive and accurate way to navigate the complexities of modern information systems.

2. PRELIMINARIES

Definition 2.1: [1]

For a nonempty set X and a set of parameters E, a pair (G, E) is said to be a soft set over X provided that G is a map of E into the power set P(X).

i.e., (G, E) = $\{(e,(e)): e\in E \text{ and } (e) \in P(X)\}$.

Each G (e) is called a component of (G, E) (or approximation of e) and a family of all soft sets defined over X with E is denoted by S (X_E). It is clear that the number of soft sets in S (X_E) is $2^{|X||E|}$. We use the notation G_E for the soft set instead of (G, E).

Definition 2.2: [1]

A soft set (G,E) over X is said to be subset of a soft set (H,E) over X if $G(e)\subseteq H(e)$ for each $e\in E$ and is denoted by(G,E)⊆(H,E). The soft sets (G, E) and (H, E) are said to be soft equal if each of them is a subset of the other.

Definition 2.3: [1]

A soft set (G, E) over X is said to be null soft set if all the components are the empty set and it is denoted by $\emptyset$.

Definition 2.4: [1]

A soft set (G, E) over X is said to be absolute soft set if all the components are the initial set X and it is denoted by $\tilde{X}$.

Definition 2.5: [1]

The relative complement of a soft set (G, E) is a soft set ${G_E}^C$ where G^C is a map of E into the power set P(X) defined by $G^C(e)=X-G(e)$ for all e $\in E$.

Definition 2.6: [6]

A union of two soft sets (F, A) and (G, B) over a common universe X is the soft set (H, C) where C = AUB and $\forall e \in C$,

$$H(e)=\begin{cases} \text{F(e) if e} \in \text{A} - \text{B} \\ G(e)\ if\ e \in \text{B} - \text{A} \\ F(e) \cup \text{G(e) if e} \in \text{A} \cap \text{B} \end{cases}$$

It is denoted by (F, A) ∪ (G,B)=(H,C).

Definition 2.7: [6]

An intersection of two soft sets (F,A) and (G,B) over a common universe X is the soft set (H, C), where C = A ∩ B and ∀ $e\in C$, H$(e)=F(e) \cap G(e)$.

It is denoted by (F,A) ∩̃ (G,B)=(H,C).

Definition 2.8: [6]

Let (F, A) and (G, B) be two soft sets over the common universe X. Then the Cartesian

product of (F, A) and (G, B) is defined as (F, A) ×(G, B) = (H, A× B), where H: $A \times B \rightarrow P(X \times X)$ and H(a, b) = F(a) × G(b), where (a, b) ∈ $A \times B$. i.e., H(a,b)={(h_i,h_j): $h_i \in F(a) and\ h_j \in G(b)$}

Definition 2.9: [6]

A soft set in which all the components are equal is called stable soft set. i.e., for a soft set (F, A) with universal set X and parameter set A, F(a) = Y ∀ a∈A where Y is any subset of X.

Definition 2.10: [6]

A soft set (F,A)over X is said to be

(i) A full soft set provided that $\cup_{a\in A}(a)=X$

(ii) A partition soft set provided that the family {F(a):$a\in A$} constitutes a partition of Y.

Notations:

In this paper, We denote soft sets by Greek letters β, γ, μ, η where each β = (F_β,E) is a soft set over the Universe Set U with parameter Set E. The symbol X, Y denote subsets of U. This notation is used consistently throughout the manuscript.

3. SOFT TOPOLOGY

Definition 3.1: [6]

Let $\tilde{\tau}$ be the collection of soft sets over X. Then $\tilde{\tau}$ is called a soft topology on X, if $\tilde{\tau}$ satisfies the following.

(i) $\emptyset, X \in \tilde{\tau}$

(ii) The union of an arbitrary number of soft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$.

(iii) The intersection of a finite number of soft sets $\tilde{\tau}$ belongs to $\tilde{\tau}$.

The triplet (X, $\tilde{\tau}$, E) is called a soft topological space over X and the members of $\tilde{\tau}$ are said to be soft open sets in X.

Definition 3.2: [6]

A soft topology τ on X is said to be an Enriched soft topology if τ contains all soft open sets G_E such that G(e) = $\phi\ or\ X\ \forall\ e\in E$.

Definition 3.3: [6]

(X, τ, E) is said to be stable if every soft open set is stable.

Definition3.4: [6]

Let (X, τ, E) be a soft topological space and $\phi \neq Y \subseteq X$. A family τ_Y={$Y\cap G_E$:$G_E\in\tau$} is called a soft relative topology on Y and the triple (Y,τ_Y,E) is called a soft subspace of (X, τ, E).

Proposition 3.1: [6]

Let (X, τ, A) be a soft topological space. Then τ_a= {β (a):$\beta_A\in\tau$} is a classical topology on X for each a∈A. τ_a is called a parametric topology and (X, τa) is called a parametric topological space.

Proof:

Let a ∈A be arbitrary and τ_a= {β (a):$\beta_A\in\tau$}. We have to prove that τ_a is a topology on X. Since $\emptyset, X \in \tau$, ϕ, X ∈ τ_a. Now, let $V = \cup_{a\in A}$ β (a). Since union of any number of soft open sets is soft open, V is one of the approximation in the soft open set which is the union of all soft open sets in τ. This implies that $V\in\tau_a$. Now, let U= β (a_1)∩β(a_2). Since intersection of any two soft

open sets is soft open, U is one of the approximation in the soft open set which is the intersection of two soft open sets that approximates a_1 and a_2 to the subsets $\beta(a_1)$ and $\beta(a_2)$ of X. This implies that, $U \in \tau a$. Hence $\tau a = \{\beta(a) : \beta_A \in \tau\}$ is a topology on X.

Theorem 3.1: [6]

Let (Y, τ_Y, E) be a soft sub space of (X, τ, E). Then λ_E is a soft closed set in (Y, τ_Y, E) if and only if there is a soft closed set μ_E of (X,τ,E) such that $\lambda_E = Y \tilde{\cap} \mu_E$.

Proof:

Let λ_E be a soft closed set in (Y, τ_Y, E). λ_E is a soft closed set in (Y, τ_Y, E) iff the relative complement of λ_E is open in (Y, τ_Y, E). [That is, ${\lambda_E}^C$ is open in (Y,τ_Y,E) where ${\lambda_E}^C : E \to P(Y)$ defined by ${\lambda_E}^C(e) = Y - \lambda_E(e) \forall e \in E$] iff ${\lambda_E}^C = Y \tilde{\cap} \beta_E$ [by definition 3.4] where β_E is open in (X,τ,E) iff ${\lambda_E}^C(e) = Y \cap \beta_E(e)\ \forall\ e \in E$ iff $Y - \lambda_E(e) = Y \cap \beta(e)\ \forall\ e \in E$ iff $Y - (Y \cap \beta_E(e)) = \lambda_E(e)\ \forall\ e \in E$ iff $Y - (\beta_E(e) \cap Y) = \lambda_E(e)\ \forall\ e \in E$ iff $Y - \beta(e) = \lambda_E(e)\ \forall e \in E$ iff ${\beta_E}^C(e) \cap Y = \lambda_E(e) \forall e \in E$. [That is, $\lambda_E : E \to P(Y)$ such that $\lambda_E(e) = {\beta_E}^C(e) \cap Y$] iff $\lambda_E = Y \tilde{\cap} {\beta_E}^C$ iff $\lambda_E = Y \tilde{\cap} \mu_E$ where $\mu_E = {\beta_E}^C$ is closed in (X, τ,E) [Since β_E is open in (X, τ, E), ${\beta_E}^C$ is closed in (X, τ, E)]. Hence λ_E is closed in (Y, τ_Y, E) iff $\lambda_E = Y \tilde{\cap} \mu_E$.

4. PARTIAL BELONG, PARTIAL NON-BELONG, TOTAL BELONG AND TOTAL NON-BELONG RELATIONS IN SOFT SET THEORY

Definition 4.1: [1]

Let G_E be a soft set over X and x ∈X. We have the following relations:

(i) $x \in G_E$ if $x \in G(e)\ \forall e \in E$. We read it as x totally belongs to G_E or G_E totally contains x.

(ii) $x \notin G_E$ if $x \notin G(e)$ for some $e \in E$. We read it as x does not partially belong to G_E or G_E does not partially contain x.

(iii) $x \Subset G_E$ if $x \in G(e)$ for some $e \in E$. We read it as x partially belongs to G_E or G_E partially contains x.

(iv) $x \Subset\!\!\!\!/\ G_E$ if $x \notin G(e)$ for each $e \in E$. We read it as x does not totally belong to G_E or G_E does not totally contain x.

Definition 4.2: [6]

Let (X, τ, E) be a soft topological space. Then (X, τ, E) is said to be

(i) Soft T_0 space if for every $x \neq y$, there is a soft open set U_E such that $x \in U_E$ and $y \notin U_E$ or $y \in U_E$ and $x \notin U_E$.

(ii) Soft T_1 space if for every $x \neq y$, there are two soft open sets U_E, V_E such that $x \in U_E$ and $y \notin U_E$; and $y \in V_E$ and $x \notin U_E$.

(iii) Soft T_2 space if for every $x \neq y$, there are two disjoint soft open set U_E, V_E such that $\in U_E$ and $y \notin U_E$; and $y \in V_E$ and $x \notin V_E$.

(iv) Soft regular space if for every soft closed set H_E and $x \in X$ such that $x \notin H_E$, there are two disjoint soft open sets U_E, V_E such that $H_E \subseteq U_E$ and $x \in$.

(v) Soft normal space if for every two disjoint soft closed sets H_E, F_E, there are two disjoint soft open sets U_E, V_E such that $H_E \subseteq U_E$ and $F_E \subseteq V_E$.

(vi) Soft T_3 space if it is both soft regular and soft T_1 space.

(vii) Soft T_4 space if it is both soft normal and soft T_1 space.

5. W-SOFT T_i SPACES, i=0 to 4

Definition 5.1: [6]

Let (X, τ, E) be a soft topological space. Then (X, τ, E) is said to be

a. A w-soft T_0 space if for every two distinct points $x \neq y$, there is a soft open sets U_E such that $x \Subset U_E$ and $y \notin U_E$; or $y \Subset U_E$ and $x \notin U_E$.

b. A w-soft T_1 space if for every $x \neq y$, there are two soft open sets U_E, V_E such that $x \Subset U_E$ and $y \notin U_E$ and $y \Subset V_E$ and $x \notin V_E$.

c. A w-soft T_2 space if for every $x \neq y$, there are two disjoint soft open sets U_E, V_E such that $x \Subset U_E$ and $y \notin U_E$ and $y \Subset V_E$ and $x \notin U_E$.

d. A w-soft regular space if for every soft closed set H and $x \in X$ such that $x \notin H_E$, there are two disjoint soft open sets U_E, V_E such that $H_E \subseteq U_E$ and $x \Subset V_E$.

e. A w-soft normal space if for every two disjoint soft closed sets H_E, F_E, there are two disjoint soft open sets U_E, such that $H_E \subseteq U_E$ and $F_E \subseteq V_E$.

f. A w-soft T_3 space if it is both w-soft regular and w-soft T_1 space.

g. A w-soft T_4 space if it is both w-soft normal and w-soft T_1 space.

Proposition 5.1: [6]

(i) Every w-soft T_i space (X, τ, E) is a w-soft T_{i-1} space for i=1,2,3.

(ii) Every soft T_i space (X, τ, E) is a w-soft T_i space for i=0, 1,2,3,4.

Proof:

(i) In the cases i=1,2, the proof follows from (i),(ii),(iii) of Definition 5.1.

For the case i =3,

Let $x \neq y$ in a w-soft T_3 space (X, τ, E). Since it is w-soft T_1, there exists two soft open sets β_E, γ_E such that $x \Subset \beta_E$ and $y \notin \beta_E$; and $y \Subset \gamma_E$ and $x \notin \gamma_E$. Now $x \notin \beta_E^{C}$. Since (X, τ, E) is w-soft regular, there exists two disjoint soft open sets δ_E, μ_E such that $\beta_E^{C} \subseteq \delta_E$ and $x \Subset \mu_E$. This automatically implies that $y \Subset \delta_E$ and $y \notin \mu_E$. Thus (X, τ, E) is a w-soft T_2 space.

(ii) Let (X, τ, E) be a soft T_i space, i = 0, 1, 2, 3, 4. For the case i = 0, let $x \neq y$ in a soft T_0 space (X, τ, E). Since (X, τ, E) is a soft T_0 space, there exists soft open set β_E such that $x \in \beta_E$ and

$y \notin \beta_E$; or $y \in \beta_E$ and $x \notin \beta_E$. This implies that $x \Subset \beta_E$ and $y \notin \beta_E$ or $y \Subset \beta_E$ and $x \notin \beta_E$. Hence (X, τ, E) is a w-soft T_0 space.

For the case i=1,

Let $x \neq y$ in a soft T_1 space (X, τ, E). Since (X, τ, E) is a soft T_1 space, there exists two soft open sets β_E, γ_E such that $x \in \beta_E$ and $y \notin \beta_E$; and $y \in \gamma_E$ and $x \notin \gamma_E$. This implies that $x \Subset \beta_E$ and $y \notin \beta_E$; and $y \Subset \gamma_E$ and $x \notin \gamma_E$. Hence (X, τ, E) is a w-soft T_1 space.

For the case i=2,

Let $x \neq y$ in a soft T_2 space (X, τ, E). Since (X, τ, E) is a soft T_2 space, there exists disjoint soft open sets β_E, γ_E such that $x \in \beta_E$ and $y \notin \beta_E$; and $y \in \gamma_E$ and $x \notin \gamma_E$. This implies that there exists two disjoint soft open sets β_E, γ_E such that $x \Subset \beta_E$ and $y \notin \beta_E$; and $y \Subset \gamma_E$ and $x \notin \gamma_E$. Hence (X, τ, E) is a w-soft T_2 space.

For the case i=3,

Let $x \in X$ and β_E be any closed set not containing x. Since (X, τ, E) is a soft T_3 space, there exists disjoint open sets δ_E, μ_E such that $x \in \delta_E$ and $\beta_E \subseteq \mu_E$. This implies that there exists disjoint open sets δ_E, μ_E such that $x \Subset \delta_E$ and $\beta_E \subseteq \mu_E$. Hence (X, τ, E) is w-soft T_3 space.

For the case i = 4, Since (X, τ, E) is a soft $_4$ space, it follows from the definition of w-soft T_4 space.

The following example shows that the above proposition cannot be reversed.

Example 5.1

Consider the following soft sets defined over the universal set Y = {y, z} with the set of parameters E = {a, e} as follows:

$$\beta_{1_A}=\{(a,\{y\}),(e,Y)\}$$

$$\beta_{2_A}=\{(a,\{y\}),(e,\{z\})\}$$

$$\beta_{3_A}=\{(a,Y),(e,\emptyset)\}$$

$$\beta_{4_A}=\{(a,\emptyset),(e,Y)\}$$

$$\beta_{5_A}=\{(a,Y),(e,\{y\})\}$$

$$\beta_{6_A}=\{(a,\emptyset),(e,\{y\})\}$$

Then the collections

$\eta_1=\{\phi,Y,\beta_1\}$,

$\eta_2=\{\phi,Y,\beta_{2_A}\}$,

$\eta_3=\{\phi,Y,\beta i_A : i=3,4,5,6\}$ are soft topologies on Y.

For $y \neq z$, it can be noted that

(i) $y \Subset \beta_{1_A}$ and $z \notin \beta_{1_A}$. But any soft open set of (Y, η_1,E) partially containing z contains y as well. So that (Y, η_1, E) is a w-soft T_0 space but not w-soft T_1 pace.

(ii) The soft open set β_{2_A} of (Y, η_2, E) satisfies a condition of a w-soft T_1 space. On the other hand, there are no two disjoint soft open sets such that one of them partially contains y and the other partially contains z. So that (Y, η_2,E) is not a w-soft T_2 space.

(iii) The soft open sets β_{3_A} and β_{4_A} of (Y, η_3,E) are disjoint such that $y \Subset \beta_{3_A}$ and $z \notin \beta_{3_A}$;

and that $z \Subset \beta_{4_A}$ and $y \notin \beta_{4_A}$. Also, any two disjoint soft closed sets of (Y, η_3, E) are separated by two disjoint soft open sets. Hence (Y, η_3, E) is a w- soft T_2 space and w-soft T_4 space. On the other hand ${\beta_{6_A}}^C$ is a soft closed set such that $y \notin {\beta_{6_A}}^C$. Since the only soft open sets of (Y, η_3, E) containing ${\beta_{6_A}}^C$ is the absolute soft set, (Y, η_3, E) is a not w-soft T_3.

Remarks 5.1

(i) In the above example, we show that (Y, η_3, E) is a w-soft T_2 space. It is noted that (Y, η_3, E) is a enriched soft topological space. It is also true for all enriched soft topological space. Hence we conclude that every enriched soft topological space is a w-soft T_2 space.

(ii) It is well known that the coincidence between soft topology and classical topology occurs when E is a singleton set, that is w-soft T_i coincide with T_i spaces. Also, in the above example we show that (Y, η_3,E) is a w-soft T_4 space, but it is not w-soft T_3. Hence the concepts of w-soft T_3 spaces and w-soft T_4 spaces are independent from each other.

(iii) Every w-soft T_1 space contains at least three soft open sets.

(iv) Every w-soft T_2 space contains at least four soft open sets.

Proposition 5.2: [6]

If β_E is a soft closed open set in (Y,η,E) such that β_E and its relative complement are full soft sets, then (Y, η, E) is a w-soft T_1.

Proof:

Let $y \neq z \in Y$. Since β_E is a full soft set, $y \Subset \beta_E$ and $\boldsymbol{z} \Subset \beta_E$. This implies that $\Subset \beta_E$ and $z \notin \beta_E$; and since ${\beta_E}^C$is a full soft set, $\Subset {\beta_E}^C$ and $z \Subset {\beta_E}^C$. This implies that $y \notin {\beta_E}^C$and $z \Subset {\beta_E}^C$. By Hypothesis, is a soft open set. Hence (Y, η, E) is a w- soft T_1 space.

Proposition 5.3: [6]

If β_E is a soft closed open set in (Y, η, E) such that β_E and its relative complement are full soft sets, then (Y, η, E) is a w-soft T_2.

Proof:

Let $y \neq z \in Y$. Since β_E is a full soft set, $y \Subset \beta_E$ and $z \Subset \beta_E$. This implies that $y \Subset \beta_E$ and $z \notin \beta_E$; and since ${\beta_E}^C$ is a full soft set, $\Subset {\beta_E}^C$ and $z \Subset {\beta_E}^C$. This implies that $y \notin {\beta_E}^C$and $z \Subset {\beta_E}^C$. By Hypothesis, is a soft open set. Therefore, β_E and ${\beta_E}^C$are two disjoint soft open sets such that $y \Subset \beta_E$ and $z \notin \beta_E$; and $y \notin {\beta_E}^C$and $z \Subset {\beta_E}^C$. Hence (Y, η, E) is a w-soft T_2 space.

Theorem 5.1: [6]

Let (Y,η,E) be a w-soft regular space. Then the following statements are equivalent:

(i) (Y,η,E) is a w-softT_2

(ii) (Y,η,E) is a w-softT_1

(iii) (Y,η,E) is a w-soft T_0

Proof:

The implications (i)⇒(ii)⇒ (iii) follows from proposition 5.1. For (iii) ⇒ (i), let Y≠z in a w-soft T_0 space (Y,η,E). Then there exists a soft open set β_E such that $y \Subset \beta_E$ and $z \notin \beta_E$; or $z \Subset \beta_E$ and $y \notin \beta_E$. Consider $y \Subset \beta_E$ and $z \notin \beta_E$. Now $y \Subset \beta$ implies that $y \Subset \beta_E^{\,C}$ implies that $y \notin \beta_E^{\,C}$. Now, $\Subset \beta_E$ ⇒ $z \Subset \beta_E^{\,C}$. Since $\beta_E^{\,C}$ is a soft closed set of a w-soft regular space (Y, η, E) not containing y, there exists two disjoint soft open sets δ_E, μ_E such that $y \Subset \delta_E$ and $z \Subset \beta_E^{\,C} \subseteq \mu_E$. Obviously, $z \notin \delta_E$, since δ_E and μ_E are disjoint. Also, $\Subset \beta_E^{\,C} \subseteq \mu_E$ and $y \notin \mu_E$, since $y \notin \beta_E^{\,C}$ and $\beta_E^{\,C} \subseteq \mu_E$. Hence (Y,η,E) is a w-soft T_2.

Theorem 5.2: [6]

The parametric topological space (Y, τ_e) is T_i iff (Y, τ, E) is a w-soft T_i space in the cases of i = 0, 1.

Proof:

Let i=1 and $y \neq z \in Y$. Suppose that (Y, τ_e) is T_1. Then τ_e contains two open sets V, W such that y ∈ V, $z \notin$ V; and $z \in W$,$\notin W$. This implies that τ contains two soft open sets β_E, μ_E such that $(e)=V$ and $(e)= W$. It can be seen that $y \Subset \beta_E$, $z \notin \beta_E$ and $z \Subset \mu_E$, $y \notin \mu_E$. Thus, (Y, τ, E) is a w-soft T_1. The other case is proved similarly.

The Sufficient part of this theorem can be proved simply by retracing the steps in the Necessary part.

The following example shows that Theorem5.2 does not holds for i=2,3,4

Example 5.2:

Consider the following soft sets defined over the universal set Y = {y, z} with the set of parameters E = {a, e} as follows:

$\beta_{1_A}=\{(a,\{z\}),(e,\emptyset)\}$

$\beta_{2_A}=\{(a,\{z\}),(e,\{y\})\}$

$\beta_{3_A}=\{(a,Y),(e,\{z\})\}$

Then (Y, η, E)=$\{\tilde{\phi},Y,\beta_{1_A},\beta_{2_A},\beta_{3_A}\}$ is a soft topology on Y. Now (Y, η_e) = $\{\phi, Y,\{y\},\{z\}\}$ is T_2,T_3 and T_4. However (Y, η, E) is not w-soft (i = 2, 3, 4), since the soft open sets of η are not disjoint.

Theorem 5.3: [6]

If (Y,τ,E) is stable such that all (Y, τ_e) are T_i iff (Y,τ,E) is w-soft T_i in the cases of i = 0, 1, 2, 3, 4

Proof:

The result follows from the Theorem 5.2 for the case i=0,1.

For the case i=2,

Let $y \neq z \in Y$ and suppose that (Y, τ_e) is T_2. Then τ_e contains two disjoint open sets V, W such that $y \in V$, $\notin V$; and $z \in W$, $y \notin W$. This

implies that τ contains two disjoint soft open sets $\beta_E, \boldsymbol{\mu}_E$ [This is possible, since (Y,η,E) is stable] such that $\beta(e)=V\ \forall\ e\in E$ and $\beta\ (e)=W\ \forall e\in E$. It can be seen that y$\Subset\beta_E, \notin\beta_E$ and $z\Subset\mu_E, y\notin\mu_E$. Thus, (Y,τ,E) is a w-soft T_2.

For the case i=3,

Let $x \in X$ and β_E be any closed set not containing x. Then x $\notin \beta\ (e)\ \forall e \in E$, since (Y, η, E) is stable. Now, Since (Y, τ_e) is T_3 space, τ_e contains two disjoint open sets V, W such that x$\in V\ and\ \beta\ (e)\subseteq W\forall e\in E$. This implies that there exists disjoint open sets δ_E, μ_E [This is possible, since (Y, η, E) is stable] such that $x \Subset \delta_E$ and $\beta_E\subseteq \mu_E$. Thus, (Y, τ, E) is a w-soft regular. Also, by theorem 5.2, (Y, τ, E) is a w-soft T_1. Hence (Y, τ, E) is a w-soft T_3 space.

For the case i=4,

Let β_E and γ_E be two disjoint closed sets in (Y,η,E). Then $(e)\neq(e)\forall e\in E$. Since (Y, τ_e) is T_4 space, τ_e contains two disjoint open sets V,W such that $\beta\ (e)\subseteq\ V\ \forall e\ \in E\ and\ (e)\subseteq W\ \forall e\in E$, Since (Y,η,E) is stable. This implies that there exists disjoint open sets δ_E, μ_E [This is possible, since (Y,η,E) is stable] such that $\beta_E\subseteq\ \delta_E$ and $\gamma_E\subseteq\ \mu_E$. Thus, (Y, τ, E) is a w-soft normal. Also, by theorem 5.2, (Y, τ, E) is a w-soft T_1. Hence (Y, τ, E) is a w-soft T_4 space.

The Sufficient part of this theorem can be proved simply by retracing the steps in the Necessary part.

Proposition 5.4: [6]

Every soft subspace (X, η_X, E) of a w-soft T_i space (Y, η, E) is a w-soft T_i in the cases of i = 0, 1, 2, 3.

Proof:

Let i = 3. First we prove that (X, η_X, E) is a w-soft T_1. Suppose $y \neq z \in Y$. Since (Y,η, E) is a w-soft T_1, η contains two soft open sets β_E and γ_E such that $y\Subset \beta_E, z\notin \beta_E$; and $z\Subset\gamma E, y\notin\gamma_E$. So $\delta_E=\tilde{X}\sqcap\beta_E$ and $\mu_E=\tilde{X}\sqcap\gamma_E$ are soft open sets of (X, η_X, E) such that $y\Subset\ \delta_E, z\notin\delta_E$; and $z\Subset\mu_E, y\notin\mu_E$. Thus, (X, η_X, E) is a w-soft T_1. To prove the w-soft regularity of (X, η_X, E), let ρ_E be a soft closed set of (X, η_X, E) such that $y\in X$ and $y\notin\rho_E$. Then η contains a soft closed set λ_E such that $\rho_E=\tilde{X}\sqcap\lambda_E$ and y $\notin\lambda_E$. So η contains two disjoint soft open sets ζ_E, κ_E such that $\lambda_E\subseteq\zeta_E$ and $\boldsymbol{y} \Subset \kappa_E$. Now, $W_1=\tilde{X}\sqcap\zeta_E$ and $W_2=\tilde{X}\sqcap\kappa_E$ are two disjoint soft open subsets of (X, η_X, E) such that $\rho_E\subseteq W_1$ and $y\Subset W_2$. Thus, (X, η_X, E) is a w-soft regular. Hence (X, η_X, E) is w-soft T_3. The other cases are proved similarly.

Theorem 5.4: [6]

The finite product of w-soft T_i spaces is a w-soft T_i space in the cases of i=0,1,2

Proof:

Let i=2. Without loss of generality, we consider two w-soft T_2 spaces $(X,,_1)$ and (Y,τ,E_2). Suppose that $(x_1,\ y_1)\neq(x_2,y_2)$ in $(X\times Y,\ \eta\times\tau,$

$E_1 \times E_2$). Then $x_1 \neq x_2$ *or* $y_1 \neq y_2$. Let $x_1 \neq x_2$. Then η contains two disjoint soft open sets β_E and γ_E such that $x_1 \Subset \beta_E$, $x_2 \notin \beta_E$; and $x_2 \Subset \gamma_E$, $x_1 \notin \gamma_E$. Now $\beta_E \times Y$ and $\gamma_E \times Y$ are soft open sets in $\eta \times \tau$ such that $(x_1, y_1) \Subset \beta_E \times Y$ and $(x_2, y_2) \notin \beta_E \times Y$; and $(x_2, y_2) \Subset \gamma_E \times Y$ and $(x_1, y_1) \notin \gamma_E \times Y$. Since $(\beta_E \times X) \cap (\gamma_E \times Y) = \emptyset_{E_1 \times E_2}$, $(X \times Y, \eta \times \tau, E_1 \times E_2)$ is a w- soft T_2 space. The other cases are proved similarly.

6. APPLICATIONS

This chapter illustrates how partial belong relation is applied in decision making problems. We formulate one real example using soft set and apply the concept of partial belong relation in order to make correct decision (Table 1).

Table 1: Comparative Analysis of the Proposed Implementation with Existing Frameworks.

ASPECT	CLASSICAL METHODS	CURRENT METHODOLOGY
Classification of Work	Conventional Methodologies	Partial belonging relation methodology
Novelty	Real life Applications	A novel Mathematical frame work
Application	General Problem solving and Decision Making	Selection and Ranking of alternatives based on the Partial Belonging Relation Approach
Data Type	Qualitative Approach	Quantitative and measurable variables
Contribution of the Extended Work	Focuses on the Partial Belonging Relation and its fundamental properties	Establishes and validates theoretical propositions deriving applications in diverse domains
Outcome of the Research	Ranking the items and identifying the optimal solution	Ranking alternatives to determine the most preferred choice

STEPS TO FIND OPTIMAL DECISION USING PARTIAL BELONG RELATION

- Formulate the given real life problem into mathematical model using soft sets.
- Classify the elements in the universal set, U based on the categories α in the parameter set for each parameter set E_i as a soft set β_{E_i} over U.
- Find a value for each $y_i \in U$ and $(\alpha)$by the rule: If $y_i \Subset \beta_{E_i}$, then assign $(y_i, \beta_{E_i}(\alpha))=1$, otherwise $(y_i, \beta_E(\alpha))=0$.
- Now reduce the each parameter set E_i to the set F_i based on the desired choice and repeat step 3 for the reduced parameter set F.
- For each $y_i \in U$, calculate the total weight as as follows: Total Weight, $T_i = \sum (y_i, \gamma(\alpha))$ where γ_{F_i} is a soft set over U with parameter set F_i.
- Determine an optimal choice using the formula $d = \max\{T_i : i = 1,2,\cdots,m\}$
- If there are more than one value for d, then select any one of them

REAL LIFE EXAMPLE

Suppose we had 12 investments choices such as Fixed deposit, Savings account, Public provident fund, Equity linked savings scheme, Equity mutual fund, Debt. Mutual fund, Liquid mutual fund, Balanced mutual fund, Bonds, Stocks, Gold/Silver, Real Estate and each investment choice has factors such as Risk

tolerance, Investment Horizon, Return expectations, Liquidity needs, Inflation impact, Tax implications and each factories categorized into various types as follows

- Risk Tolerance={Low, Medium, High}
- Investment Horizon={Short term, Medium term, Long term}
- Financial Goals={Capital Preservation, Income generation, Wealth Creation, Specific goal (Retirement or Education)}
- Return Expectations={Low, Moderate, High}
- Liquidity Needs={High, Medium, Low}
- Inflation Impact={Inflation-linked returns needed, Impact acceptable, Minimal impact acceptable}
- Tax Implications={Tax optimized, Some tax impact, Tax optimization not priority}

Our task is to find the optimal investment choice for the two types of investors: Aggressive and Conservative investors based on their preference in each factor of the investment.

Preference of Conservative investors

- Risk Tolerance={Low, Medium}
- Investment Horizon={Short term, Medium term}
- Financial Goals={Capital Preservation, Income generation}
- Return Expectations={Low, Moderate}
- Liquidity Needs={High, Medium}
- Inflation Impact={Minimal impact acceptable, Inflation-linked returns needed}
- Tax Implications={Tax optimized, Some tax impact}

Preference of Aggressive investors

- Risk Tolerance={ High}
- Investment Horizon={Medium term, Long term}
- Financial Goals={Wealth Creation, Specific goal (Retirement or Education)}
- Return Expectations={Moderate, High}
- Liquidity Needs={Medium, Low}
- Inflation Impact = {Impact acceptable, Minimal Impact acceptable}
- Tax Implications={Some tax impact, Tax optimization not priority}

Formulation of above real life situation into a Mathematical model using soft set

We take Universal set, U as the various types of investment choices and Parameter sets, $E_i$as the factors of investment. So we have 7 parameter sets and 1 universal set which are given by.

Universal set

U = {I_1 =Fixed deposit, I_2=Savings account, I_3 =Public provident fund, I_4= Equity linked savings scheme, I_5 = Equity mutual fund, I_6 =Debt. Mutual fund, I_7 = Liquid mutual fund, I_8 =Balanced mutual fund, I_9 =Bonds, I_{10} = Stocks, I_{11} = Gold/Silver, I_{12}= Real Estate}

Parameter sets

- Risk Tolerance=E_1={e_{11}=Low, e_{12}=Medium, e_{13}=High}
- Investment Horizon=E_2={e_{21}=Short term, e_{22}=Medium term, e_{23}=Long term}
- Financial Goals=E_3={e_{31}=Capital Preservation, e_{32}=Income generation, e_{33}=Wealth Creation, e_{34}=Specific goal (Retirement or Education)}
- Return Expectations=E_4={e_{41}=Low, e_{42}=Moderate, e_{43}=High}
- Liquidity Needs = E_5={e_{51}=High, e_{52}=Medium, e_{53}=Low}
- Inflation Impact = E_6= {e_{61} =Inflation-linked returns needed, e_{62} =Impact acceptable, e_{63}= Minimal impact acceptable, }
- Tax Implications = E_7= {e_{71}=Tax optimized, e_{72}= Some tax impact, e_{73}= Tax optimization not priority}

In order to find the optimal choice of investment for the two types of investors, we have to classify the investment choices based on the categories in the parameter set for each parameter set as a soft set β_{E_i}over U, i=1 to 7

For the parameter Risk Tolerance (E_1), $\beta_{E_1}(e_{11})$={I_1,I_9}; $\beta_{E_1}(e_{12})$={I_8,I_6}; $\beta_{E_1}(e_{13})$={I_{10},I_5}

For the parameter Investment Horizon (E_2), $\beta_{E_2}(e_{21})$={I_2,I_7}; $\beta_{E_2}(e_{22})$ ={I_1,I_6};

$\beta_{E_2}(e_{23})$={I_5,I_3}

For the parameter Financial Goals (E_3), $\beta_{E_3}(e_{31})$={I_3,I_1}; $\beta_{E_3}(e_{32})$ ={I_{10},I_6}; $\beta_{E_3}(e_{33})$ ={I_{10},I_5}; $\beta_{E_3}(e_{34})$={I_4,I_9}

For the parameter Return Expectations (E_4), $\beta_{E_4}(e_{41})$={I_2,I_1}; $\beta_{E_4}(e_{42})$={I_6,I_8};

$\beta_{E_4}(e_{43})$={I_5,I_{10}}

For the parameter Liquidity Needs (E_5), $\beta_{E_5}(e_{51})$={I_2,I_7};$\beta_{E_5}(e_{52})$={I_1,I_6};

$\beta_{E_5}(e_{53})$={I_3,I_4}

For the parameter Inflation Impact (E_6), $\beta_{E_6}(e_{61})$ ={I_{11},I_{12}}; $\beta_{E_6}(e_{62})$={I_{10},I_5}; m $\beta_{E_6}(e_{63})$={I_1,I_2}

For the parameter Tax Implications (E_7), $\beta_{E_7}(e_{71})$={I_3,I_4}; $\beta_{E_7}(e_{72})$={I_5,I_{10}}; $\beta_{E_7}(e_{73})$={I_{11},I_{12}}

Now using partial belong relation, For each $I_i \in U$ and $\alpha \in E_i$, Define

$$(I_i, \alpha) = \begin{cases} 1 & if\ I_i \in \beta_{E_i}(\alpha) \\ 0 & if\ I_i \notin (\alpha) \end{cases}$$

Using this relation we forma Table 2 which shows the distribution of investment choices according to E_i.

Table 2: Distribution of investment choices according to E_i

I_i	E_1			E_2			E_3				E_4			E_5			E_6			E_7		
	e_{11}	e_{12}	e_{13}	e_{21}	e_{22}	e_{23}	e_{31}	e_{32}	e_{33}	e_{34}	e_{41}	e_{42}	e_{43}	e_{51}	e_{52}	e_{53}	e_{61}	e_{62}	e_{63}	e_{71}	e_{72}	e_{73}
I_1	1	0	0	0	1	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0
I_2	0	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	0	0	0
I_3	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0
I_4	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	1	0	0
I_5	0	0	1	0	0	1	0	0	1	0	0	0	1	0	0	0	0	1	0	0	1	0
I_6	0	1	0	0	1	0	0	1	0	0	0	1	0	0	1	0	0	0	0	0	0	0
I_7	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
I_8	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
I_9	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
I_{10}	0	0	1	0	0	0	0	1	1	0	0	0	1	0	0	0	0	1	0	0	1	0
I_{11}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1
I_{12}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1

Here $I_i, = 1\ to\ 12$, denotes the various types of investment and e_{ij} denotes the category in the Parameter set E_i.

Reduction of E_i to F_i based on Conservative investors Preferences

- Risk Tolerance=F_1={e_{11}=Low, e_{12}= Medium}
- Investment Horizon=F_2={e_{21}=Short term, e_{22}=Medium term}
- Financial Goals=F_3={e_{31}=Capital Preservation, e_{32}=Income generation}
- Return Expectations=F_4={e_{41}=Low, e_{42}=Moderate}
- Liquidity Needs = F_5 = {e_{51} = High, e_{52} = Medium}
- Inflation Impact=F_6={e_{61}= Inflation-linked returns needed, e_{63}=Minimal impact acceptable, }
- Tax Implications=F_7={e_{71}=Tax optimized, e_{72}=Some tax impact}

Reduction of E_i to G_i based on Aggressive investors Preferences

- Risk Tolerance=G_1={e_{13}=High}
- Investment Horizon=G_2={e_{22}=Medium term, e_{23}=Long term}
- Financial Goals=G_3={e_{33}=Wealth Creation, e_{34}=Specific goal (Retirement or Education}
- Return Expectations=G_4={e_{42}=Moderate, e_{43}=High}
- Liquidity Needs= G_5={e_{52}=Medium, e_{53}=Low}
- Inflation Impact = G_6={e_{62}= Impact acceptable, e_{63}= Minimal Impact acceptable}
- Tax Implications=G_7={e_{72}=Some tax impact, e_{73}=Tax optimization not priority}

Table 3: Distribution of investment choices according to Conservative Investors

I_i	F_1		F_2		F_3		F_4		F_5		F_6		F_7		Total Weight $=\sum_i(I_i,\beta_{F_i}(\alpha))$
	e_{11}	e_{12}	e_{21}	e_{22}	e_{31}	e_{32}	e_{41}	e_{42}	e_{51}	e_{52}	e_{61}	e_{63}	e_{71}	e_{72}	
$\boldsymbol{I_1}$	1	0	0	1	1	0	1	0	0	1	0	1	0	0	6
I_2	0	0	1	0	0	0	1	0	1	0	0	1	0	0	4
I_3	0	0	0	0	1	0	0	0	0	0	0	0	1	0	2
I_4	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
I_5	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
I_6	0	1	0	1	0	1	0	1	0	1	0	0	0	0	5
I_7	0	0	1	0	0	0	0	0	1	0	0	0	0	0	2
I_8	0	1	0	0	0	0	0	1	0	0	0	0	0	0	2
I_9	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1
I_{10}	0	0	0	0	0	1	0	0	0	0	0	0	0	1	2
I_{11}	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1
I_{12}	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1

Table 4: Distribution of investment choices according to Aggressive Investors

I_i	G_1	G_2		G_3		G_4		G_5		G_6		G_7		Total Weight $=\sum_i(I_i,\beta_{G_i}(\alpha))$
	e_{13}	e_{22}	e_{23}	e_{33}	e_{34}	e_{42}	e_{43}	e_{52}	e_{53}	e_{62}	e_{63}	e_{72}	e_{73}	
I_1	0	1	0	0	0	0	0	1	0	0	1	0	0	3
I_2	0	0	0	0	0	0	0	0	0	0	1	0	0	1
I_3	0	0	1	0	0	0	0	0	1	0	0	0	0	2
I_4	0	0	0	0	1	0	0	0	1	0	0	0	0	2
$\boldsymbol{I_5}$	1	0	1	1	0	0	1	0	0	1	0	1	0	6
I_6	0	1	0	0	0	1	0	1	0	0	0	0	0	3
I_7	0	0	0	0	0	0	0	0	0	0	0	0	0	0
I_8	0	0	0	0	0	1	0	0	0	0	0	0	0	1
I_9	0	0	0	0	1	0	0	0	0	0	0	0	0	1
I_{10}	1	0	0	1	0	0	1	0	0	1	0	1	0	5
I_{11}	0	0	0	0	0	0	0	0	0	0	0	0	1	1
I_{12}	0	0	0	0	0	0	0	0	0	0	0	0	1	1

Determining optimal investment choice for two types of Investors: Conservative and Aggressive

The maximum total weight according to Table 3 (That is, for conservative investors) is 6 which corresponds to the investment choice I_1- Fixed deposit. Hence the optimal investment choice for conservative investors is Fixed deposit.

The maximum total weight according to Table 4 (That is, for Aggressive investors) is 6 which corresponds to the investment choice I_5– Equity Mutual fund. Hence the optimal investment choice for aggressive investors is Equity Mutual fund.

Table 5: Contemporary Applications of Partial Belong Relations Across Diverse Disciplines.

Application Area	Partial Belong Relation Represents
Medical Diagnosis	Degree of disease likelihood
Artificial Intelligence	Confidence in classification
Weather Forecasting	Degree of weather conditions
Banking	Customer creditworthiness
E-Commerce	Customer interest in product categories
Autonomous vehicles	Confidence in obstacle detection
Smart appliances	Degree of dirtiness or comfort level
Human Resources	Candidate suitability
Graph Theory	Strength of vertices and edges in fuzzy graphs
Social Media	Degree of community membership
Robotics	Distance and obstacle assessment
Agriculture	Soil moisture and crop condition

Analytical Interpretation

The above table provides a concise summary of the progressive advancement and interdisciplinary applications of partial belong relations, highlighting their growing significance across contemporary scientific and technological domains.

Partial belong relations provide a powerful mathematical framework for modeling uncertainty, ambiguity, and gradual transitions encountered in real-world systems. They are fundamental to fuzzy set theory, enabling more realistic decision-making than traditional binary relations. Today, they are widely used in artificial intelligence, healthcare, banking, robotics, autonomous vehicles, smart appliances, agriculture, social network analysis and fuzzy graph theory, where objects or entities often belong to categories to varying degrees rather than absolutely.

CONCLUSION

This paper has undertaken a comprehensive exploration of soft sets, soft topology and the novel concept of w-soft topology, generalizing the understanding of topological structures in an uncertain environment. The introduction of partial belong, partial non-belong, total belong and total non-belong relations has provided a gradient frame work for analyzing relationships between soft sets, enabling a deeper understanding of their properties and applications.

The application of these relations to w-soft separation axioms has yielded insightful results, shedding light on the topological characteristics of soft sets. Furthermore, the real life example illustrating the optimal investment choice for conservative and aggressive investors using partial belong relation underscores the practical relevance of these theoretical constructs.

we have established several propositions and theorems, thereby providing greater clarity and extending the concepts presented in the existing literature. We have also introduced new definitions and successfully applied them to derive and validate the corresponding theorems. The findings presented in this study contribute to a deeper understanding of the proposed concepts and provide a foundation for further research in this area.

The findings of this paper contribute to the

emerging field of soft topology, offering new paths for research and application in areas such as decision making in supply-chain processes, optimization of engineering design processes and uncertainty modeling in financial risk assessment. As researchers continue to explore and expand upon these concepts, the potential for impactful applications in diverse fields is substantial.

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