



ON CERTAIN STRUCTURAL PROPERTIES OF σ -NEAR RINGS

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ABSTRACT

In this paper, we investigate several structural properties of σ -near-rings under different algebraic conditions. We first examine the behaviour of σ -near-rings under anti-homomorphic mappings and establish a corresponding structural property. For regular σ -near-rings, we prove that they are reduced, and that every idempotent element belongs to the centre. We further establish an equivalence between the absence of non-zero nilpotent elements and the representation of a σ -near-ring as a subdirect product of integral near-rings. Under the additional assumption that every N -subgroup is invariant, we obtain results concerning N -subgroups, left ideals, and completely prime ideals. The results obtained contribute to the understanding of the behaviour of σ -type identities in near-ring theory and provide directions for further investigation.

Keywords: σ -Near Ring, Near-Ring, Commutativity, Ideals, Anti-homomorphism, Zero Symmetric Near-Ring

1. INTRODUCTION

In recent years, several special identities have been introduced to understand the internal structure and commutativity properties of near-rings. One such structure is the σ -near ring,

introduced by V. Alies Anbukani and co-authors. A near ring N is said to be a σ -near ring if it satisfies $x^2y = xyx$ for all $x, y \in N$. Motivated by these developments, the study of σ -near rings has attracted attention

in understanding how this identity interacts with other algebraic conditions. Various properties concerning commutativity, ideals, homomorphic images, and structural decompositions have been investigated in the literature. However, several aspects of σ -near rings remain open for further exploration, particularly when they are considered together with additional algebraic conditions such as reducedness, insertion of factors property (IFP), or special mappings like anti-homomorphisms.

Throughout this paper N stands for a right near-ring $(N, +, \cdot)$ with at least two elements and '0' denotes the identity element of the group $(N, +)$ and obviously $0n = 0$ for all $n \in N$.

2. Preliminaries

Definition 2.1

A **right near ring** is a non-empty set N together with two binary operations "+" and " $\cdot$ " such that

- (i) $(N, +)$ is a group (not necessarily abelian)
- (ii) $(N, \cdot)$ is a semigroup
- (iii) $(n_1 + n_2)n_3 = n_1n_3 + n_2n_3$ for all $n_1, n_2, n_3 \in N$

Definition 2.2

A near-ring N is said to be **zero-symmetric** if $a0 = 0$ for all $a \in N$.

Definition 2.3

N is said to fulfill the **Insertion of Factors Property (IFP)** provided that for all $a, b, n \in N$, $ab = 0 \Rightarrow anb = 0$.

Definition 2.4

A near-ring N is said to have **$(*, IFP)$** if $ab = 0$ implies $ba = 0$ and satisfy IFP.

Definition 2.5

A near-ring N is said to be **reduced** if it has no nonzero nilpotent elements.

Notation 2.6

- a) E denotes the set of all idempotent of N
(e in N is called an **idempotent** if $e^2 = e$)
- b) L denotes the set of all nilpotent of N
(a in N is **nilpotent** if $a^k = 0$ for some positive integer k)

Result 2.7 [1]

A near ring N has no non-zero nilpotent elements (i.e.) $L = \{0\}$ if and only if $x^2 = 0 \Rightarrow x = 0$ for every $x \in N$.

Result 2.8 [4]

If N has no non-zero nilpotent elements (i.e.) $L = \{0\}$ and $= N_0$, then for any a, b in N , $abe = aeb$ where e is idempotent in N and has **$(*, IFP)$** .

Definition 2.9

Let N, M be near ring then we say, a mapping $\phi: N \rightarrow M$ is called **anti-homomorphism** of near-ring if for all $a, b \in N$, $\phi(a + b) = \phi(b) + \phi(a)$ and $\phi(ab) = \phi(b)\phi(a)$.

Definition 2.10

The **centre** of a near-ring N is $C(N) = \{x \in N \mid xa = ax \text{ for all } a \in N\}$.

Definition 2.11

A near ring N is said to be **S' - near ring** according as $x \in xN$ for all $x \in N$.

Definition 2.12

A subgroup of $(M, +)$ of $(N, +)$ is called a **right N -subgroup** of N if $NM \subseteq M$. Simply it is called **N -subgroup** of N .

Definition 2.13

A subgroup M of the additive group $(N, +)$ is called an **invariant N -subgroup** of N if $NM \subseteq M$ and $MN \subseteq M$.

Definition 2.14

An ideal I of N is said to be **completely prime ideal** if for all $x, y \in N$, $xy \in I \Rightarrow x \in I$ or $y \in I$.

Result 2.15 [6]

Let N be an σ -near ring with identity. Then N is commutative.

Definition 2.16

A non-empty subset I of N is called an **ideal** of N if:

1. $(I, +)$ is a normal subgroup of $(N, +)$,
2. $IN \subseteq I$ (right ideal)
3. $a(b + i) - ab \in I$ for all $a, b \in N$ and $i \in I$. (left ideal)

Result 2.17 [1]

Let N be a zero symmetric. Then the following are equivalent:

- i) N has no non zero nilpotent elements (i.e.) $L = \{0\}$
- ii) N is a subdirect product of integral near rings.

Definition 2.18

A near ring N is said to be a **right regular near-ring** if for all $x \in N$ there exist $a \in N$ such that $a = x^2a = xax$.

Definition 2.19

A near-ring N is said to have the **property $P(m, r)$** if $x^r N = Nx^m$ for all x in N where r and m are positive integers.

Definition 2.20

Let I be an ideal of N . A near ring N is said to have **Property P_4** if for every a, b in N , $ab \in I \Rightarrow ba \in I$.

Definition 2.21

A near-ring N is called a **integral near-ring** if N has set no non-zero divisors.

Definition 2.22

The near rings N is said to be **Boolean** if $x^2 = x$ for each $x \in N$.

Example 2.23

We considered the near-ring $(N, +, \cdot)$ where $(N, +)$ is the Klein's four group with $N = \{0, a, b, c\}$ here ' $\cdot$ ' is defined as below. Then N is a σ near-ring.

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	0	b
c	0	a	0	c

Lemma 2.24 [6]

Every σ -near ring is zero symmetric.

Definition 2.25

A near ring N is said to be **commutative** if for every x, y in N , $xy = yx$.

3. MAIN RESULTS**Theorem 3.1**

Let N be a σ -near ring and f is an anti-homomorphic function of N then it satisfies $yx^2 = xyx$ for all $x, y \in M$

Proof

Let N, M represent two near rings.

Let $f: N \rightarrow M$ be a surjective anti-homomorphism

Let $a, b \in N$, since N is a σ -near ring we get $a^2b = aba$

Applying f on both sides,

$$f(a^2b) = f(aba)$$

Since f is anti-homomorphic,

$$f(a^2b) = f(b)f(a^2) = f(b)f(a)^2$$

Again consider,

$$f(aba) = f(a)f(ab) = f(a)f(b)f(a)$$

By equating we get,

$$f(b)^2f(a) = f(a)f(b)f(a)$$

Then for $x, y \in M$ we have $x = f(a)$ and

$$y = f(b)$$

Hence $yx^2 = xyx$ for all $x, y \in M$.

Theorem 3.2

Let N be a regular σ -near ring, then

- i) N has no non zero nilpotent elements
- ii) N has $(*, IFP)$

- iii) Every idempotent element of N is a centre.

Proof

Let N be a regular σ -near ring

Then for any $x \in N$ there exists an element $y \in N$

$$\text{we have } x = x^2y = xyx \quad \rightarrow (1)$$

- i) Let us assume $x^2 = 0$ in equation (1)

$$\text{Then } x = x^2y = (0)y = 0$$

Hence N is reduced, by result 2.7

We conclude that there is no nonzero nilpotent element in N .

- ii) Since σ -near ring is always zero symmetric and from i) we have $L = \{0\}$,

It directly follows from the result 2.8,

Hence N has $(*, IFP)$.

- iii) Let $e \in E$

$$\text{By equation (1), we get } e = e^2y = eye$$

$$\Rightarrow ey = eye$$

$$\text{Clearly } ye(ye - eye) = 0 \text{ and } eye(ye - eye) = 0$$

Consider

$$(ye - eye)^2 = (ye - eye)(ye - eye)$$

$$= ye(ye - eye) - eye(ye - eye) = 0$$

Since N is reduced, $(ye - eye) = 0$

$$\text{Hence } ye = eye = ey$$

Therefore, every idempotent element is a centre in N .

Remark 3.3

Whenever N is a regular σ -near ring it is clear that N is right regular. Since for all $x \in N$ there exist $y \in N$ such that $x = x^2y = xyx$.

Theorem 3.4

Let N be a σ -near ring. Then the following are equivalent:

- i) N has no non zero nilpotent elements (i.e.) $L = \{0\}$
- ii) N is a subdirect product of integral near rings.

Proof

Since every σ -near ring is zero symmetric.

By Result 2.17,

We get if N is zero symmetric. Then the following are equivalent:

- i) N has no non zero nilpotent elements (i.e.) $L = \{0\}$
- ii) N is a subdirect product of integral near rings.

Hence the equivalence holds for σ -near ring.

Theorem 3.5

Let N be a regular σ -near ring and if every N -subgroup is invariant then

- i) $A = \sqrt{A}$ where A is N -subgroup of N
- ii) Every left ideal N is an ideal
- iii) Every ideal of N is completely prime

Proof

Let N be a regular σ -near ring, then by remark 3.3 N is right regular.

Then for any $x \in N$ there exists an element $y \in N$

$$\text{we have } x = x^2y = xyx \quad \rightarrow (1)$$

- i) Let A be an N -subgroup of N and $a \in \sqrt{A}$, where $\sqrt{A} = \{x \in N \mid x^k \in A\}$ for some positive integer k .

For all $x \in N$

By equation (1), $a = axa = a^2x$

$$a = a(a^2x)x$$

$$= a^3x^2$$

$$= a^2(a^2x)x^2$$

$$= a^4x^3 \dots = a^kx^{k-1}$$

Since every N -subgroup is invariant,

$$a = a^kx^{k-1} \in AN \subseteq A.$$

- ii) Let A be a left ideal of N

Since N is zero symmetric, $NA \subseteq A$

Also give that every N -subgroup is invariant.

Hence A become an ideal of N .

- iii) Let I be an ideal of N and $a^2 \in I$

Then by equation (1),

$$a = axa = a^2x \in IN \subseteq I$$

Hence $a \in I$

Therefore, every ideal of N is completely prime.

Theorem 3.6

Let N be a σ -near ring with identity then N satisfy property P_4 .

Proof

Since N is a σ -near ring with identity, then

By result 2.15, N is always commutative.

Let I be an ideal of N

Assume $ab \in I$, since commutative it is clear that $ba \in I$

Hence N satisfies property P_4 .

Theorem 3.7

Let N be a σ -near ring then N is a $P(2,1)$ near ring.

Proof

Let $x \in N$, then $a^2x \in a^2N$

Since $a^2x = axa$

For $a \in N$, $(ax)a \in Na$

Hence $a^2N \subseteq Na$

Similarly,

we trace back the steps for $Na \subseteq a^2N$

Therefore $a^2N = Na$, hence N is a $P(2,1)$ near ring.

Theorem 3.8

Let N be a σ -near ring that has left cancellation then N is commutative.

Proof

Since N is a σ -near ring,

For all $a, x \in N$ we have $a^2x = axa$

$\Rightarrow aax = axa$

Using left cancellation, $ax = xa$

Hence N is commutative.

Theorem 3.9

Let N be a Boolean σ -near ring that has left cancellation then N is S' -near ring.

Proof

Since N is a σ -near ring,

For all $a, x \in N$ we have $a^2x = axa$

We have N is Boolean, so $ax = axa$

Using left cancellation, $x = xa$

Since $a \in N$, $x \in xN$

Hence N is S' -near ring.

CONCLUSION

In this paper, we have investigated several structural properties of σ -near-rings under different algebraic conditions. The results demonstrate that additional

conditions imposed on a σ -near-ring can lead to significant structural consequences. These observations provide a basis for further investigation of σ -near-rings and their relationships with other important classes of near-rings.

REFERENCES

1. McCoy NH (1964). *The theory of rings and near-rings*. Macmillan.
2. Pilz G (1983). *Near-rings: The theory and its applications*. North-Holland Publishing Company.
3. Kim KH and Roush FW (1987). *Near-rings and their links with groups*. Pitman Research Notes in Mathematics.
4. Clay JR and Malone DA (1992). *Near-rings and their applications*. Oxford University Press.
5. Jun Y B (2004). On insertion of factors property in near-rings. *Journal of the Chungcheong Mathematical Society*, 17(2):177–183.
6. Anbukani V A and Tamizh Chelvam T (2013). Commutativity on σ -near rings and its structures. *International Journal of Algebra*, 7(12):573–582.
7. Radha D and Lakshmi CR (2018). On weakly π -subcommutative Γ -near rings. In *Proceedings of the National Conference on*

- Innovations in Mathematics*, pp. 10–18.
8. Radha D and Lakshmi C R (2019). A study on P-weakly regular near-ring. *Journal of Emerging Technologies and Innovative Research*, 6(6):159–161.
9. Radha D, Dhivya C, Vinutha M, Muthu Maheswari K and Veronica Valli SR (2019). A study on R-near ring. *Journal of Emerging Technologies and Innovative Research*, 6(12):434–439.
10. Radha D and Dhivya C (2019). On S-near rings and S'-near rings with right bipotency. *Journal of Emerging Technologies and Innovative Research*, 6(2):952–1000.